Assignment 1.

This homework is due *Thursday*, September 6.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

1. Quick cheat-sheet

Reminder. On the set $\mathbb{R}$ of real numbers there two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) $(a+b)+c = a+(b+c)$ for all $a, b, c \in \mathbb{R}$,
(A2) $a+b = b+a$ for all $a, b \in \mathbb{R}$,
(A3) there exists $0 \in \mathbb{R}$ s.t. $0+a = a+0 = a$ for all $a \in \mathbb{R}$,
(A4) for each $a \in \mathbb{R}$ there exists an element $-a$ s.t. $a+(-a) = (-a)+a = 0$,
(M1) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$,
(M2) $ab = ba$ for all $a, b \in \mathbb{R}$,
(M3) there exists $1 \in \mathbb{R}$ s.t. $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$,
(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ s.t. $a \cdot (1/a) = (1/a) \cdot a = 1$,
(D) $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$ for all $a, b, c \in \mathbb{R}$.
(NT) $1 \neq 0$.

Reminder. Let $\mathbb{A}$ be a set with two operations $+$ and $\cdot$ satisfying A1–A4, M1–M3 and D, NT. (For example, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.) The set $\mathcal{P} \subset \mathbb{A}$ is called the set of *positive elements* if

(P1) If $a, b \in \mathcal{P}$, then $a+b \in \mathbb{P}$,
(P2) If $a, b \in \mathcal{P}$, then $ab \in \mathbb{P}$,
(P3) If $a \in \mathbb{A}$, then exactly one of the following holds: $a \in \mathcal{P}$, $a = 0$, $-a \in \mathcal{P}$.

Then we say $a < b$ if and only if $b - a \in \mathcal{P}$; $a \leq b$ if and only if $b - a \in \mathcal{P} \cup 0$.

2. Exercises

(1) (Exercise 1.1.1 in Royden–Fitzpatrick) For $a \neq 0$ and $b \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$. (Hint: check that $a^{-1}b^{-1}$ satisfies definition of $(ab)^{-1}$.)

(2) (1.1.2) Verify the following:
   (a) For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.
   (b) For each positive number a, its multiplicative inverse a^{-1} also is positive.
   (c) If $a > b$, then
   $$ac > bc \text{ if } c > 0 \quad \text{and} \quad ac < bc \text{ if } c < 0.$$
   (Hint: determine whether $ac - bc \in \mathcal{P}$.)

— see next page —

(3) In each case below, determine if P is a set of positive elements (i.e. if P satisfies P1–P3).
  (a) $\mathbb{A} = \mathbb{Z}$, $P = \mathbb{N}$,
  (b) $\mathbb{A} = \mathbb{Z}$, $P = -\mathbb{N}$,
  (c) $\mathbb{A} = \mathbb{Q}$, $P = \{r \in \mathbb{Q} : \ r > 1\}$,
  (d) $\mathbb{A} = \mathbb{C}$, $P = \{z = x + iy \in \mathbb{C} : \ x > 0\}$,
  (e) Prove that for $\mathbb{A} = \mathbb{C}$, there is no set of positive elements. (In other words, one cannot imbue $\mathbb{C}$ with a meaningful order.)

(4) (1.1.4) Let a, b be real numbers.
  (a) Show that if $ab = 0$ the $a = 0$ or $b = 0$. (Hint: multiply ab by a^{-1}.)
  (b) Verify that $a^2 - b^2 = (a - b)(a + b)$ and conclude that from part (a) that if $a^2 = b^2$, then $a = b$ or $a = -b$.
  (c) Let c be a positive real number. Define $E = \{x \in \mathbb{R} \mid x^2 < c\}$. Verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (b) to how that there is a unique $x > 0$ for which $x^2 = c$. It is denoted $\sqrt{c}$.

(5) (1.1.7+) The *absolute value* $|x|$ of a real number x is defined to be $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. For real numbers a, b verify the following:
  (a) $|ab| = |a||b|$.
  (b) (Triangle inequality) $|a + b| \leq |a| + |b|$.
  (c) (Triangle inequality) $|a - b| \geq ||a| - |b||$.
  (d) For $\varepsilon > 0$,
$$|x - a| < \varepsilon \text{ if and only if } a - \varepsilon < x < a + \varepsilon.$$

(6) (1.2.12) Problem 4c (together with in-class proposition about $\sqrt{2}$) proves existence of at least one irrational number (*irrational* means "real but not rational"). Granted that at least one irrational number exists, prove that irrational numbers are dense in $\mathbb{R}$.

3. Extra exercises

Problems below will only go to the numerator of your grade for this homework. Also, the due date on these problems is December, 7. That is, you can submit these problems any time before classes end.

(7) Give an example of a family $\mathcal{F}$ of distinct subsets of a countable set s.t. the following two conditions hold:
  - $\mathcal{F}$ is uncountable.
  - $\mathcal{F}$ is a *chain* with respect to set inclusion, i.e. for every two subsets A, B in the family $\mathcal{F}$, either $A \subseteq B$ or $B \subseteq A$.

(8) Give an example of a family $\mathcal{F}$ of distinct subsets of a countable set s.t. the following two conditions hold:
  - $\mathcal{F}$ is uncountable.
  - For any $A, B \in \mathcal{F}$, the intersection $A \cap B$ is *finite*.